Theoretical Mechanics
Mid-Term Solution

1. The Lagrangian for the cart is

$$L_C = \frac{m}{2}\dot{x}^2 - \frac{k}{2}(x)^2$$

Because the bob is attached to the cart, the Lagrangian for the bob is

$$L_B = \frac{M}{2}\left(\dot{x} + L\dot{\theta}\right)^2 + MgL\cos\theta$$

a. Neglecting inessential constants, in the small oscillation approximation the total Lagrangian is

$$L = L_C + L_B = \frac{m}{2}\dot{x}^2 + \frac{M}{2}\left(\dot{x} + L\dot{\theta}\right)^2 - \frac{k}{2}x^2 - \frac{MgL}{2}\theta^2$$

The equations of motion are

$$m\ddot{x} + M\left(\ddot{x} + L\ddot{\theta}\right) + kx = 0$$

$$M\left(\ddot{x} + L\ddot{\theta}\right) + MgL\theta = 0$$

Taking the second variable to be $L\theta$, the mass matrix is

$$\underline{M} = \begin{pmatrix} m+M & M \\ M & M \end{pmatrix}$$

and the $\underline{K}$ matrix is

$$\underline{K} = \begin{pmatrix} k & 0 \\ 0 & Mg \end{pmatrix}$$

b. The normal mode frequencies solve

$$\det\left(\underline{K} - \omega^2\underline{M}\right) = 0 \rightarrow \det\begin{pmatrix} k - \omega^2(m+M) & -\omega^2 M \\ -\omega^2 M & Mg - \omega^2 M \end{pmatrix} = 0$$

$$\det\begin{pmatrix} 2-\omega^2 2 & -\omega^2 \\ -\omega^2 & 1-\omega^2 \end{pmatrix} = 0 \rightarrow 2\left(1-\omega^2\right)^2 - \omega^4 = 0$$

$$\omega^4 - 4\omega^2 + 2 = 0 \qquad \omega^2 = \frac{4 \pm \sqrt{16-8}}{2} \qquad \omega^2 = 2 \pm \sqrt{2}$$

c. The plus normal mode has

$$\begin{pmatrix} -2-2\sqrt{2} & -2-\sqrt{2} \\ -2-\sqrt{2} & -1-\sqrt{2} \end{pmatrix}\begin{pmatrix} x \\ L\theta \end{pmatrix} = 0 = -1-\sqrt{2}\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} x \\ L\theta \end{pmatrix}$$

The bob moves antisymmetric to the cart with an amplitude $\sqrt{2}$ times larger.
The minus normal mode has

$$\begin{pmatrix} -2+2\sqrt{2} & -2+\sqrt{2} \\ -2+\sqrt{2} & -1+\sqrt{2} \end{pmatrix}\begin{pmatrix} x \\ L\theta \end{pmatrix} = 0 = -1+\sqrt{2}\begin{pmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} x \\ L\theta \end{pmatrix}$$

The bob moves symmetric with the cart with an amplitude $\sqrt{2}$ times larger.

2. As mentioned several times in lectures, a good way to understand the magnetic field is in terms of the magnetic field form

$$\omega_{\vec{B}}^2 = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy,$$

where $\vec{B}$ is the usual magnetic field "vector".

a. How is the Maxwell Equation $\nabla \cdot \vec{B} = 0$ expressed in terms of the exterior derivative?

$$d\omega_{\vec{B}}^2 = \left(\nabla \cdot \vec{B}\right) dx \wedge dy \wedge dz = 0$$

Therefore, the magnetic field form is a closed form.

b. Suppose γ is closed curve and σ_1 and σ_2, two non-intersecting surfaces with $\partial\sigma_1 = \partial\sigma_2 = \gamma$. Show, using generalized Stoke's Theorem

$$\int_{\sigma_1} \omega_{\vec{B}}^2 = \int_{\sigma_2} \omega_{\vec{B}}^2.$$

In other words, the magnetic flux through a closed loop is independent of the surface used to compute it.

Let V be the volume enclosed by the two surfaces. Then $\partial V = \sigma_1 - \sigma_2$, where it is assumed that the surface normal for σ_1 points out of the volume and the surface normal for σ_2 points into the volume. By the generalized Stoke's Theorem

$$0 = \int_V d\omega_{\vec{B}}^2 = \int_{\sigma_1 - \sigma_2} \omega_{\vec{B}}^2 = \int_{\sigma_1} \omega_{\vec{B}}^2 - \int_{\sigma_2} \omega_{\vec{B}}^2 \rightarrow \int_{\sigma_1} \omega_{\vec{B}}^2 = \int_{\sigma_2} \omega_{\vec{B}}^2.$$

c. If a magnetic vector potential $\vec{A}$ is found such that $\vec{B} = \nabla \times \vec{A}$, what is $\int_\gamma \omega_{\vec{A}}^1$?

Again we can apply generalized Stoke's Theorem

$$\int_{\sigma_1} \omega_{\vec{B}}^2 = \int_{\sigma_2} \omega_{\vec{B}}^2 = \int_{\sigma_1, \sigma_2} \omega_{\nabla \times \vec{A}}^2 = \int_{\sigma_1, \sigma_2} d\omega_{\vec{A}}^1 = \int_\gamma \omega_{\vec{A}}^1.$$

3. The Lagrangian for the one dimensional motion of a particle in a uniform gravitational is

$$L = \frac{m}{2} v_y^2 - mgy,$$

where y is a vertical coordinate and $v_y = dy/dt$.

a. Show the Hamiltonian is

$$H(y, p) = \frac{p^2}{2m} + mgy,$$

$$p = \frac{\partial L}{\partial \dot{y}} = mv_y$$

$$H(y, p) = p\frac{p}{m} - L = p\frac{p}{m} - \frac{p^2}{2m} + mgy = \frac{p^2}{2m} + mgy,$$

b. Show the Hamiltonian equations of motion give the usual Newtonian equation for a uniformly accelerating motion.

$$\dot{y} = \frac{\partial H}{\partial p} = \frac{p}{m} = v_y \quad \dot{p} = -\frac{\partial H}{\partial y} = -mg \rightarrow \ddot{y} = \frac{\dot{p}}{m} = -g$$

c. Is the Hamiltonian explicitly dependent on time? No
What is the Hamilton-Jacobi equation for this problem? The action solves

$$\frac{1}{2m}\left(\frac{\partial S}{\partial y}\right)^2 + mgy = \alpha \rightarrow S\left(y,\alpha,t\right) = \pm\sqrt{2m}\int\sqrt{\alpha - mgy}dy - \alpha t$$

d. Solve the Hamilton-Jacobi equation for the action function $S\left(y,\alpha,t\right)$. Let $\beta = \partial S / \partial\alpha$. Solve for $y = y\left(\alpha,\beta\right)$.

$$S\left(y,\alpha\right) = \pm\frac{\sqrt{2m}}{-mg}\frac{2\left(\alpha - mgy\right)^{3/2}}{3} - \alpha t \rightarrow \beta = \pm\frac{\sqrt{2m}}{-mg}\left(\alpha - mgy\right)^{1/2} - t$$

$$\alpha - mgy = \frac{\left(\beta + t\right)^2 mg^2}{2} \rightarrow y = \frac{\alpha}{mg} - \frac{\left(\beta + t\right)^2 g}{2}$$

e. Show the initial conditions applied to the solution in d. yield the constants of the motion

$$\begin{aligned} \alpha &= \frac{m}{2}v_0^2 + mgy_0 \\ \beta &= -\frac{v_0}{g} \end{aligned}.$$

and the usual equation for uniform acceleration.

$$y_0 = \frac{\alpha}{mg} - \frac{\beta^2 g}{2} \qquad v_0 = \left.\frac{dy}{dt}\right|_0 = -\beta g$$

$$\therefore \alpha = \frac{mv_0^2}{2} + mgy_0 \qquad y\left(t\right) = y_0 + v_0 t - \frac{t^2}{2}g$$